Rogue waves in discrete-time quantum walks

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Rogue waves are rapid and unpredictable events of exceptional amplitude reported in various fields, such as oceanography and optics, with much of the interest being targeted towards their physical origins and likelihood of occurrence. Here, we use the all-round framework of discrete-time quantum walks to study the onset of those events due to a random phase modulation, unveiling its long-tailed statistics, survival time, and dependence upon the degree of randomness. We find the minimal disorder strength allowing for their occurrence to scale $\propto N^{-1/2}$, N being the number of sites. Moreover, an extreme-value analysis converges to the Gumbel class of limiting distributions.

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I. INTRODUCTION

Rogue or freak waves, unpredictable and rare huge walls of water appearing from nowhere and vanishing without a trace, have been known and feared for centuries by seafarers. The first solid account of the phenomenon took place in 1995 when data collected on the Draupner oil platform in the North Sea revealed a 26-m wave rising out of a background with about half significant wave height [1]. Years later, analogies between such ocean wave phenomena and light propagation in optical fibers surged in the framework of the nonlinear Schrödinger equation [2]. Since then, interest in ubiquitous wave phenomena displaying long-tailed statistics, when outliers occur more often than expected from Gaussian statistics, has skyrocketed in various fields (for a recent review, see Ref. [3]). Optics, particularly, has been a powerful test bed for investigating rogue waves owing to the spatial and timescales involved and, in addition, optical rogue waves include an assortment of novel phenomena, not necessarily featuring a hydrodynamics counterpart [3].

One of the key challenges in the field is to find out precisely how those events emerge so as to be able to predict and control them. There is a long-standing debate on whether rogue waves are primarily driven by linear or nonlinear processes [4] and what is the role of noise and randomness [5]. It is natural to assume that nonlinearity plays an important role due to modulational instability [6,7], collisions between solitons [8], and so forth. On the other hand, some studies suggest that the linear interference of random fields is crucial [9–20], with nonlinear effects responsible for extra wave focusing [21–23]. Indeed, linear models can display rogue waves on their own when augmented with the right ingredients as shown in Ref. [10]. This has been shown experimentally in microwave transport in randomly distributed scatterers [9], two-dimensional (2D) photonic crystal resonators [13], and very recently by measuring linear light diffraction patterns in the presence of long-range spatial memory effects in the random input [18].

Interest in linear rogue waves has been increasing considerably over the past few years. Yet, it is surprising that quantum mechanics has barely been taken into consideration. Even though the dynamics of a single quantum particle can be mapped into linear optics, investigating the onset of roguelike events in the very domain of quantum mechanics has its own appeal. It could, for instance, shed new light on the dynamics of disordered systems and related features such as Anderson localization. With that in mind, we set about to explore the occurrence of *rogue quantum amplitudes* using the discrete-time quantum walk (DTQW) approach [24]. It is basically a cellular automaton [25] whose updating rules are run by a preset sequence of quantum gates. Given recent experimental advances in the field [26–28] as well as their wide range of applications, from quantum algorithms [29] to the simulation of involved phenomena in condensed matter physics [15,30–34], DTQWs make for a suitable starting point.

We report the manifestation of rogue waves in the Hadamard one-dimensional DTQW induced by random phase fluctuations. We do so by unveiling the long-tailed statistics of the occupation probability amplitudes (which is analogous to light intensity in optics) over the space-time set of events. We show that an intermediate level of disorder scaling as $N^{-\nu}$ maximizes the likelihood of rogue events. That has to do with a fair balance between localization and mobility, for which the localization length $\propto N^{2\nu}$, N being the number of sites. Furthermore, an extreme-value analysis is carried out for the amplitude block maximum over time and we find that the resulting distribution falls into the Gumbel class.

II. QUANTUM WALK MODEL

We consider a single-particle DTQW in one dimension [24] defined by a two-level (coin) space $H_C = \{|\uparrow\rangle, |\downarrow\rangle\}$ and a position space $H_P = \{|n\rangle\}$, such that the full Hilbert space reads $H = H_C \otimes H_P$. An arbitrary state at a given instant t can be written as $|\Psi(t)\rangle = \sum_{n=1}^{N}[a_n(t)|\uparrow, n\rangle + b_n(t)|\downarrow, n\rangle]$, satisfying the normalization condition $\sum_n P_n(t) = \sum_n[|a_n(t)|^2 + |b_n(t)|^2] = 1$.

The quantum walker evolves as $|\Psi(t+1)\rangle = \hat{S}(\hat{C} \otimes I_p)\hat{D}|\Psi(t)\rangle$, where the conditional shift operator $\hat{S}$ is responsible for the nearest-neighbor transitions $\hat{S}|\uparrow, n\rangle = |\uparrow, n+1\rangle$ and $\hat{S}|\downarrow, n\rangle = |\downarrow, n-1\rangle$ (assuming periodic boundary conditions), $\hat{C} = (|\uparrow\rangle\langle\uparrow| + |\uparrow\rangle\langle\downarrow| + |\downarrow\rangle\langle\uparrow| - |\downarrow\rangle\langle\downarrow|)/\sqrt{2}$ is the standard Hadamard coin, I_p is the identity operator acting on the N-dimensional position space, and

$$\hat{D} = \sum_{c}\sum_{n} e^{iF(c,n,t)}|c, n\rangle\langle c, n| \tag{1}$$

is the phase-gain operator, with $F(c, n, t)$ being a real-valued arbitrary function [35] and $c = \uparrow, \downarrow$. Periodic boundary conditions are not essential for the generation of rogue waves and are set to filter out barrier reflections from the statistics. Given the flexibility in choosing $F(c, n, t)$, one is able to produce various dynamical regimes. Setting $F = 0$ renders the standard Hadamard quantum walk in which walker spreads out ballistically [24]. Here, instead, we set a static random phase modulation such that $F(c, n, t) = F(c, n) = 2\pi\nu$, where ν is a random number uniformly distributed within $[-W, W]$, with W being the disorder width. As this setting can lead to Anderson localization [15,32], we ought to inquire whether rogue waves can be supported given proper initial conditions and the amount of noise embedded in $F(c, n)$.

III. RESULTS

In order to avoid ambiguity between an actual rogue event (a rare one) and the inevitable Anderson localization in the statistics, we initialize the system in a coin-unbiased [36], fully delocalized state $|\psi(0)\rangle = \frac{1}{\sqrt{2N}}\sum_{n=1}^{N}(|\uparrow, n\rangle + i|\downarrow, n\rangle)$. Note that an initial localized state would only invite a few modes to act in the evolution rendering narrow periodic beatings in time rather than the unpredictable transient nature of a rogue wave. Random phase modulation is applied at every step by gate $\hat{D}$ [see Eq. (1)] so as to foster inhomogeneity and, as a result, fragmentation of the walker wave function. These two ingredients have been proved to be crucial for the development of linear rogue waves [10].

Let us now establish the criteria to identify the rogue waves. Following the standard procedure as in oceanography and optics [3,37], we define a occupation probability threshold P_{th} as the mean of the largest one third of values on a full space-time record. A rogue-wave event is counted whenever $P_n(t) > 2P_{\mathrm{th}}$.

Figure 1 shows a snapshot of a typical rogue-wave event with $P_n \approx 5P_{\mathrm{th}}$ alongside a detailed look over the amplitude record over space and time. The peak shares all the standard characteristics of a rogue wave: Besides the large amplitude in comparison to the background, it is unpredictable and short lived.

In Fig. 2 we show normalized probability density functions (PDFs) of P_n generated by a large ensemble of quantum walk runs (space-time records with $N \times$ steps values) for some representative strengths of disorder. Figure 2(b) clearly displays another key signature of the occurrence of rogue events [3], which is a positively skewed, L-shaped distribution. It features a significant number of outliers in the high-amplitude range, relatively rare among the total number of events yet more than what one would get from Gaussian statistics.

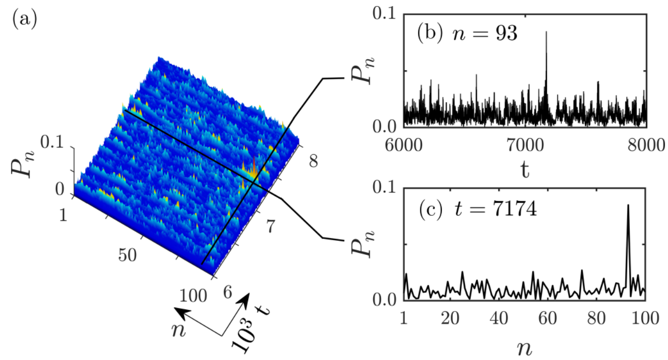

FIG. 1. (a) Snapshot of the space-time evolution of the occupation amplitude P_n in the Hadamard DTQW on a ring with $N = 100$ sites and disorder strength $W = 0.1$ (single realization). (b), (c) Time series and spatial profile extracted from (a). The rogue event is seen at $t = 7174$.

In order to analyze those distributions in a more quantitative level for the whole range of W, Fig. 3(a) shows the ensemble-averaged percentage of events fulfilling the condition $P_n > 2P_{\mathrm{th}}$. Figure 3(b) shows the disorder level W_c,

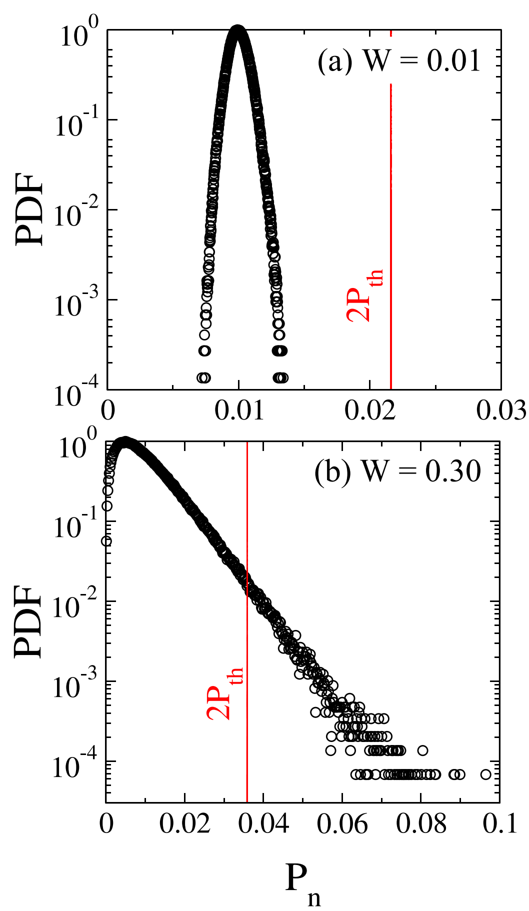

FIG. 2. Normalized PDFs for (a) $W = 0.01$ (Gaussian profile) and (b) $W = 0.3$ (exponential decay) in semilog scale for an ensemble of 5000 independent realizations of disorder and 10^4 steps on a cycle with $N = 100$ sites. The vertical line marks the threshold above which the outcome is considered a rogue event.

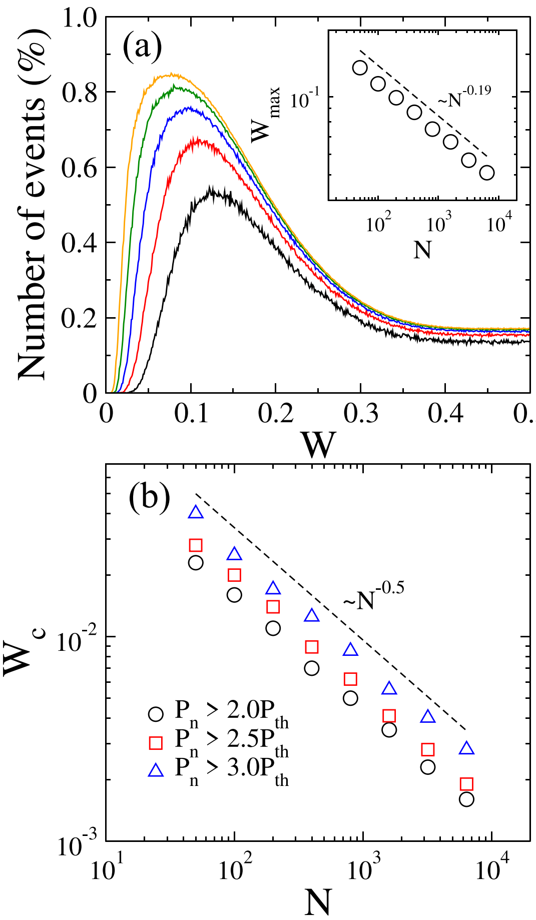

FIG. 3. (a) Number of rogue-wave events vs disorder strength W for $N = 50$, 100, 200, 400, and 800 sites [from the black (lower) line to the orange (upper) line], averaged over 5000 independent realizations of disorder, each running through $10N$ time steps. This choice is arbitrary given the evolution is statistically stationary in time. What affects the localization behavior and the rogue-wave statistics is the spatial domain N. The inset shows the scaling of the disorder degree that maximizes the chances of measuring a rogue event W_{max} with N. (b) Disorder strength W_c above which rogue waves have a finite occurrence probability for distinct threshold levels. The scaling $W_c \propto N^{-1/2}$ unveils that at W_c the localization length $\chi \propto 1/W^2$ is of the order of the chain size.

above which there is a finite chance that rogue waves take place, as a function of N. So as not to be led by the arbitrary threshold $2P_{th}$, we consider other levels as well. Irrespective of that, the minimal disorder strength leading to the occurrence of rogue waves $W_c \propto N^{-1/2}$. Such scaling behavior can be fully explained by stressing that the occurrence of rogue-wave events here is a disorder-induced phenomenon. Therefore, they set in when the associated Anderson localization length χ becomes smaller than the system size N. In the opposite regime, the system is effectively disorder free. According to the above consideration, rogue waves will appear whenever $\chi < N$. In the weak disorder regime, the typical localization length of the eigenstates of quantum walks under random phase shifts depends quadratically on the inverse of the disorder width, i.e., $\chi = k/W^2$, with k being a constant [15] (it simply saturates in the strong disorder limit). The condition for the emergence of rogue waves is thus $k/W^2 < N$, or $W > \sqrt{k/N}$, in full agreement with the scaling law reported in Fig. 3(b). We highlight that such a scaling property as well as the overall behavior seen in Fig. 3 hold for much lower number of steps and system sizes within experimental feasibility [38].

Another curious feature is the optimal level of disorder W_{max} that maximizes the chances of observing a rogue event somewhere along the system. This suggests that rogue events are more likely to develop when localization and mobility are properly balanced. The inset of Fig. 3(a) shows that $W_{max} \propto N^{-\nu}$, with $\nu \approx 0.19$ over the range of chain sizes considered. For such a disorder level, the typical localization length scales as $\chi \propto N^{2\nu}$.

An increased likelihood of the occurrence of rogue waves between weak and intermediate disorder strengths has been seen in recent experiments carried out on one-dimensional (1D) photonic lattices featuring both on-site and coupling disorder [17]. That also suggests that the interplay between localization and delocalization is a key ingredient for the generation of extreme events in linear systems. Furthermore, correlated fluctuations have been exploited to enhance the likelihood of occurrence of rogue waves [16,18], some of these largely exceeding the amplitude threshold (referred to as super rogue waves) [18].

Large fluctuations in $F(c, n)$ [cf. Eq. (1)] tend to make the localization effects sharper but it does not necessarily mean that the occurrence of rogue waves will follow that up. We shall always keep in mind that a rogue wave is a rare and sudden event whose amplitude should exceed some threshold based on the average amplitude background. In order to produce such abnormal constructive interference at some location via linear dynamics, we need the proper synchronization of random waves undergoing different paths and thereby some degree of mobility. Figure 4 shows the evolution of branching patterns highlighting the distribution profile of the rogue events (red spots). In the case of weak disorder, we note that whenever synchronization conditions are met to form a rogue wave, it usually covers a few sites in the neighborhood before disappearing [Fig. 4(a)]. For intermediate disorder, the rogue events become sparse but more frequent, as a more complex branching profile emerges [Fig. 4(b)]. If we keep on increasing the disorder width W, there will be a stage above which mobility, if any, is restricted to shorter spatial domains given the onset of local resonances. This is seen in Fig. 4(c) in the form of well-defined amplitude domains, with a few of them giving rise to rogue waves now and then. That is why the rogue-wave likelihood saturates for large W and barely responds to the system size N [see Fig. 3(a)]. Each panel of Fig. 4 also shows the normalized inverse participation ratio $\mathrm{IPR}(t) = [N \sum_n P_n(t)^2]^{-1}$ so we can get a more global view of the dynamics. Although it does not really capture the rogue-wave statistics because it reads the entire wave function at a time, the IPR ultimately tells us that long-time evolution is not necessary to observe rogue waves because it displays stationary behavior following a very rapid transient time. To confirm this property (in the wide sense) we run Dickey-Fuller tests for an IPR series covering thousands of steps, all of which resulted in the null hypothesis that a unit root is present in a first-order autoregressive model of the

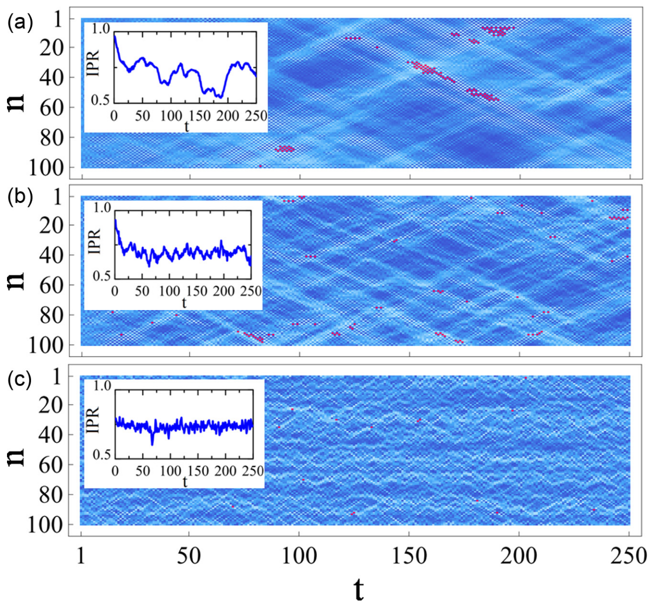

FIG. 4. Samples of space-time branching patterns of $P_n(t)$ for disorder widths (a) $W = 0.05$, (b) $W = 0.1$, and (c) $W = 0.5$. Red (darker) spots are rogue-wave events. Insets show the evolution of the corresponding normalized IPR.

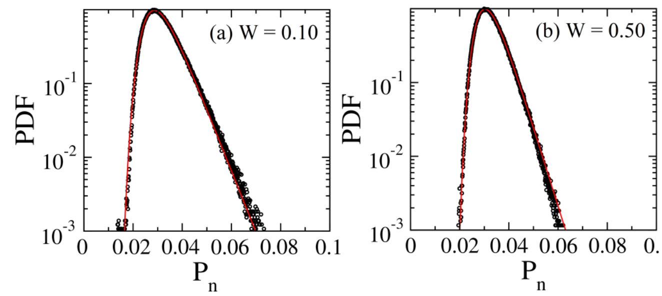

FIG. 6. Extreme-value PDFs in semilog scale for $N = 100$ and 10^4 independent realizations of disorder. At each time step, the maximum probability amplitude $P_{\max}$ is recorded. The red (solid) line is a Gumbel-type fitting given by $f(z) \propto \exp[-\alpha z - \beta \exp(-\alpha z)]$, with (α, β) being (184.73, 106.12) for (a) $W = 0.10$ and (232.51, 1002.07) for (b) $W = 0.5$.

form $y_t = \alpha_0 + \alpha_1 y_{t-1} + \epsilon_t$, the last term denoting uncorrelated noise.

We have just seen that if a rogue wave is at time t, depending on W there is a good chance another one will be observed in the surroundings at $t + 1$. To account for this curious feature, let us do the following. For a given run, we track the appearance of rogue waves at a given time and count how many steps are covered before no rogue waves are to be seen. Two or more rogue waves occurring simultaneously at distinct locations count as one elapsed time. Counting resets as soon as another occurrence is detected and so forth. We define τ as the mean of those observations. Figure 5 shows this quantity versus W for different sizes N. It immediately tells us the rogue-wave durability correlates with the likelihood of observing one [by randomly picking up an event from the ensemble; see Fig. 3(a)]. Whereas the process of rogue-wave generation here is statistically stationary in time, it is influenced by N as long as $\chi \propto W^{-2}$ is reasonably smaller for constructive interference to build up, usually involving a number of neighboring sites.

Last but not least, let us perform an extreme-value analysis by selecting the maximum amplitude at each time step during the evolution. By doing this block maxima approach, we are able to establish, for all practical purposes, a set of identically distributed random variables $\{y_1, y_2, \ldots, y_k\}$, to which the Fischer-Tippett-Gnedenko theorem applies [39,40]. In short, it states that given the cumulative distribution of $M = \max(y_1, y_2, \ldots, y_k)$, $F(y) = [\int_{-\infty}^{y} p(y')dy']^k$, it is possible to reach an asymptotic limit for a particular sequence of scaling factors a_k and b_k such that $\lim_{k\to\infty} F(a_k z + b_k) = G(z)$. The key point is that $G(z)$ comes in three distinct forms depending on the parent function $p(y)$. For the ones carrying faster-than-power-law decay (which is our situation; cf. Fig. 2), say $p(y) \sim e^{-y^{\mu}}$, $\mu > 0$, the limiting cumulative distribution is found to be $G(z) = e^{-e^{z}}$. Then, the extreme-value PDF is the Gumbel distribution $f(z) = G'(z) = e^{-z-e^{-z}}$, with $z \in (-\infty, \infty)$. Figure 6 shows that our extreme events indeed belong to the Gumbel class. For an intermediate degree of disorder, as in Fig. 6(a), the range of $P_{\max}$ is visibly more stretched, which again indicates a pronounced likelihood of observing a rogue event.

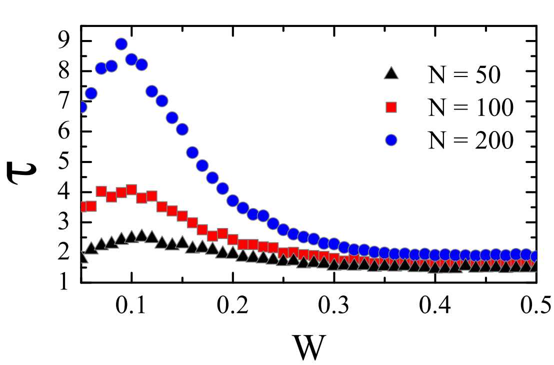

FIG. 5. Mean rogue-wave time span τ over 3000 steps vs disorder strength W, averaged over 1000 independent realizations of disorder. For intermediate values of W chances are there will be other occurrences of rogue waves nearby in subsequent steps.

IV. FINAL REMARKS

We reported rogue waves of linear origin in disordered DTQWs. Such events take place when the disorder strength breaks through $W_c \propto N^{-1/2}$. The highest likelihood of observing rogue waves is obtained for intermediate values of randomness that scale as $W_{\max} \propto N^{-\nu}$, due to a proper balance between trapping mechanisms and mobility for which the localization length $\chi \propto N^{2\nu}$. Further investigations are in order to build up on the intrinsic relationship between localization length and rogue-wave generation. We went over the long-tail profile to prove that disordered Hadamard DTQW falls within the Gumbel class of extreme-value limiting distributions.

The DTQW studied here offers the possibility of embedding nonlinearity into $F(c, n, t)$. In this context, solitonlike pulses and self-trapping have been reported in Refs. [33,35].

The stage is set for addressing the competition between linear and nonlinear mechanics in the generation of rogue waves in quantum walks.

From the experimental side, state-of-the-art photonic platforms [41] are able to perform the simulation efficiently up to several steps, especially given the quantum walk we investigated here relies on interference between single-particle states. To bypass problems with the scalability of resources, time multiplexing techniques are often employed [38,42–44]. In these, the quantum walker position is encoded in the arrival times of light pulses performing round trips through a couple of fiber loops of different lengths. Technological progress is such that those synthetic lattices are currently capable of running Bloch oscillations [45], forming solitons [43,44], and much more [46]. While noise can deteriorate the useful signal after about 100 round trips [38], the rogue waves we found here can be generated in just a few time steps as the dynamics is statistically stationary.

In addition to delivering significant advances in the field of linear rogue waves, we hope our findings seed interest in quantum-mechanical extreme events in general. This is of high value in quantum information processing, where unexpected events of that nature could lead to potential hazards during the dynamics given the unavoidable presence of noise in quantum devices.

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