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LOCALIZATION OF ACOUSTIC WAVES IN ONE-DIMENSIONAL MODELS WITH CHAOTIC ELASTICITY

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In this paper we study the propagation of acoustic waves in a one-dimensional system with nonstationary chaotic elasticity distribution. The elasticity distribution is assumed to have a power spectrum $S(f) \sim 1/f^{(2B-3)/(B-1)}$ for $B \geq 1.5$. By using a transfer-matrix method we solve the discrete version of the scalar wave equation and compute the Lyapunov exponent. In addition, we apply a second-order finite-difference method for both the time and spatial variables and study the nature of the waves that propagate in the chain. Our numerical data indicate the presence of weak localized acoustic waves for high degree of correlations ($B > 2$).

Keywords: Acoustic waves; localization; Bernoulli map.

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1. Introduction

By considering a one-electron Hamiltonian, Anderson *et al.* have predicted the absence of extended eigenstates in low-dimensional systems with uncorrelated disorder.[1] Therefore, at the long time the width of the time-dependent wave-packet saturates in a finite region around the initial position. In a three-dimensional lattice, the presence of weak disorder promotes the localization of the high-energy eigenmodes.[1,2] The prediction of exponential localization of all one-electron eigenfunctions in one-dimensional (1D) systems can be violated when special short-range[3–8] or long-range[9–11] correlations are present in the disorder distribution. From the experimental point of view, these theoretical predictions were useful to explain transport properties of semiconductor superlattices[8] and microwave transmission spectra of a single-mode waveguide with intentional correlated disorder.[11]

The localization theory applies also to the study of magnon localization in random ferromagnets,[12] collective vibrational motion of 1D disordered harmonic chains[13,14] and acoustic waves in disordered media.[15–24] In fact, the propagation of acoustic waves has attracted both theoretical[15–23] and experimental[24] interest. In general lines, it was shown that such waves may be localized in media with uncorrelated disorder. However, recent works point out the drastic effect of correlations within the acoustic waves context.[19–23] In Ref. 19 the propagation of acoustic waves in the random-dimer chain was studied using the transfer-matrix method, exact analytical analysis and direct numerical simulation of the scalar wave equation. The results indicate that there exists a resonance frequency at which the localization length of the acoustic wave diverges.[19] It was also shown that only the resonance frequency can propagate through the 1D media. Moreover, the wave propagation in random system with power-law correlation function was investigated by using renormalization group formalism as well as numerical methods.[20–23] Calculations indicate that there can be a disorder-induced transition from delocalized to localized states of acoustic waves in any spatial dimension.

In this paper we study the propagation of acoustic waves in a 1D system with nonstationary chaotic elasticity distribution. The elasticity distribution is generated following the modified Bernoulli map.[25] The map can generate a stationary and nonstationary sequence by changing a single parameter B. For $B \geq 1.5$ the sequence has a power spectrum $S(f) \sim 1/f^{(2B-3)/(B-1)}$. By using a transfer-matrix method we solve the discrete version of the scalar wave equation and compute the Lyapunov exponent. In addition, we apply a second-order finite-difference (FD) method for both the time and spatial variables and study the nature of the waves that propagate in the chain. Our numerical data indicate the presence of weak localized acoustic waves for high degree of correlations ($B > 2$).

2. Model and Formalism

Following Ref. 19, the acoustic wave equation in a random media is given by

$$\frac{\partial^2}{\partial t^2}\psi(x,t) = \frac{\partial}{\partial x}\left[\eta(x)\frac{\partial\psi(x,t)}{\partial x}\right], \tag{1}$$

where $\psi(x,t)$ is the wave amplitude, t is the time, and $\eta(x) = e(x)/m$ is the ratio of the stiffness $e(x)$ and the medium mean density m. We consider the wave amplitude with a time-dependent harmonic form $\psi(x,t) = \psi(x)\exp(-i\omega t)$, where ω is the wave frequency. We will use a FD method to write the acoustic wave equation in a discretized form. The spatial wave amplitude $\psi(x)$ is written as ψ_i where $x = i\Delta x$. The spatial derivative will be written as $(\partial\psi(x))/(\partial x) \approx (\psi_i - \psi_{i-1})/\Delta x$. Following Ref. 19 we will use $m = 1$ and consider nearest-neighbor spacing $\Delta x = 1$. Therefore, the right side of Eq. (1) can be written as

$$\frac{\partial}{\partial x}\left[\eta(x)\frac{\partial\psi(x)}{\partial x}\right] \approx [\eta_i(\psi_{i+1} - \psi_i) - \eta_{i-1}(\psi_i - \psi_{i-1})]. \tag{2}$$

Accordingly, the discrete 1D version of the wave equation can be obtained as

$$\eta_i(\psi_{i+1} - \psi_i) - \eta_{i-1}(\psi_i - \psi_{i-1}) + \omega^2\psi_i = 0. \tag{3}$$

The elastic constants η_i will be generated following the modified Bernoulli map.[25] We briefly introduce the modified Bernoulli map and the statistical properties of the sequence. The map has been introduced to investigate the basic property of the intermittent chaos and the Hamiltonian chaos by Aizawa *et al.*[26]:

$$\begin{aligned} X_{i+1} &= X_i + 2^{B-1}(1-2b)X_i^B + b, && 0 \le X_i < 0.5, \\ X_{i+1} &= X_i - 2^{B-1}(1-2b)(1-X_i)^B + b, && X_i \ge 0.5, \end{aligned} \tag{4}$$

where B is a bifurcation parameter which controls the correlation of the sequence and b stands for the small perturbation which is set as $b = 10^{-13}$ in this paper. By considering the map time scale as the number of iterations we can classify the map as stationary for $B < 2$ and nonstationary for $B > 2$ (see Ref. 25). The stationary property is recovered by the perturbation though the essential property remains invariant for a long time $i < i_b$, where $i_b \approx (2b)^{(1-B)/B}$ (see Refs. 25 and 26). In the following, we use $\eta_i = 5 + 3(X_i - \langle X_i \rangle)$. With the above procedure, the distribution of η_i has sharp edges for any value of B, which results on long-range correlated sequences of strictly positive elastic constants even when very large chains are considered.

3. Numerical Calculation

3.1. *Lyapunov exponent*

Equation (3) can be solved by using the transfer matrix formalism (TMF).[13,19] The TMF is obtained from a matrix recursive reformulation of Eq. (3). The matricial equation is

$$\begin{pmatrix} \psi_{i+1} \\ \psi_i \end{pmatrix} = \begin{pmatrix} \dfrac{-\omega^2 + \eta_i + \eta_{i-1}}{\eta_i} & -\dfrac{\eta_{i-1}}{\eta_i} \\ 1 & 0 \end{pmatrix} \begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix} = T_i \begin{pmatrix} \psi_i \\ \psi_{i-1} \end{pmatrix}. \tag{5}$$

The wave amplitude of the complete 1D system is given by the product of the transfer matrices $Q_N = \prod_{i=1}^{N} T_i$. The logarithm of the smallest eigenvalues of the limiting matrix $\Gamma = \lim_{N\to\infty}(Q_N^\dagger Q_N)^{1/2N}$ define the Lyapunov exponent Γ (inverse of localization length $\lambda = 1/\Gamma$). Typically, we use up to $N = 2^{23}$ transfer matrices to compute the Lyapunov exponent. For extended states, $\Gamma \approx 0$ and is finite for localized waves. A quantitative scaling analysis of the localization number can be derived by using the average Lyapunov exponent $\langle\Gamma\rangle$, defined as

$$\langle\Gamma\rangle = \frac{1}{N_f} \sum_{\omega=0.5}^{\omega=1.0} \Gamma(\omega), \tag{6}$$

where N_f is the number of acoustic modes within the interval [0.5, 1.0]. To compute the scaled average localization length, the bottom of the band was avoided because the localization length of these low-frequency modes are large even in the absence of correlated disorder.[19] We are interested in the existence of extended states apart from the bottom of the band. Accordingly, $\langle \Gamma \rangle$ goes to zero extended modes and is finite for localized ones. From the finite-size scaling point of view, $\langle \Gamma \rangle N \approx$ constant for extended states.

3.2. *Dynamics*

In addition, we apply the FD method with second-order discretization for both time and spatial variables proposed in Ref. 19. Thus, in discretized form, $\psi(x,t)$ is written as ψ_i^n, where n denotes the time step number and i is the grid point number.[19] Therefore, the second time derivative in Eq. (1) is given by[19]

$$\frac{\partial^2}{\partial t^2}\psi(x,t) \approx \frac{\psi_i^{n+1} - 2\psi_i^n + \psi_i^{n-1}}{\Delta t^2}, \tag{7}$$

where Δt is the size of the time step. The spatial derivative will be written as

$$\frac{\partial}{\partial x}\left[\eta(x)\frac{\partial \psi(x,t)}{\partial x}\right] \approx \frac{1}{\Delta x^2}[\eta_i(\psi_{i+1}^n - \psi_i^n) - \eta_{i-1}(\psi_i^n - \psi_{i-1}^n)]. \tag{8}$$

In our calculations the spacing Δx between two neighboring grid points was set $\Delta x = 1$. In order to ensure the stability of the discretized equations we will use $\Delta t < \Delta x/100$. We carry our dynamical analysis by sending a wave from one side of the chain ($L = 0$) and recording the transmitted wave close to the other side (position $L = 20\,000$). We calculate the intensity spectrum of the transmitted wave at position L, defined as

$$A(\omega) = \left(\frac{1}{2}\right)|\psi_L(\omega)|^2, \tag{9}$$

where $\psi_L(\omega)$ is the Fourier transform of the transmitted wave $\psi_L(t)$ at position $L = 20\,000$. For transmitted acoustic modes, $A(\omega) > 0$ and goes to zero for filtered ones. In our dynamical calculations the chain length was $N = 2^{15}$.

4. Results

In Fig. 1 we show the Lyapunov exponent Γ versus ω computed for $B = 1.5$ and 3, and system size $N = 2^{21}$. It should be stressed that the transfer-matrix method used here automatically takes care of statistical fluctuations. The resulted data have statistical errors less than 5%. We estimate and control these statistical fluctuations following the deviations of the calculated eigenvalues of two adjacent iterations.[2,19] For $B = 1.5$ the Lyapunov exponent is vanishing only for $\omega = 0$. Therefore, for $\omega > 0$, there are no truly delocalized states at this regime of weakly correlated elasticity. However, for $B = 3$, the Lyapunov exponent seems to be vanishing in a wide region of low frequencies ($\Gamma(\omega < \omega_c \approx 3) \approx 1/N$). This calculation indicates the possibility of a phase of low-frequency extended states for strongly correlated chaotic elasticity distribution. To give further informations about this trend we plot in Fig. 2 the average Lyapunov exponent $\langle \Gamma \rangle$ versus B for $N = 2^{21}$. In perfect agreements with

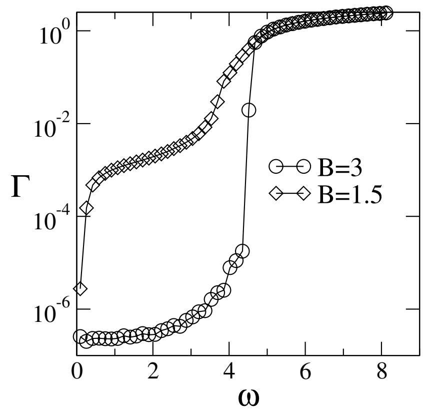

Fig. 1. Lyapunov exponent Γ versus ω computed for $B = 1.5$ and 3, and system sizes $N = 2^{21}$. For $B = 1.5$, only for $\omega = 0$, Lyapunov exponent is vanishing. For $B = 3$, it seems that the Lyapunov is vanishing for $\omega < \omega_c \approx 2$. This result suggests the possibility of a phase of low-frequency extended states for strongly correlated limit.

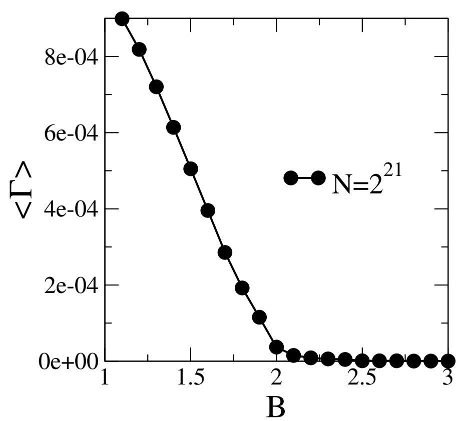

Fig. 2. Average Lyapunov exponent $\langle\Gamma\rangle$ versus B for $N = 2^{21}$. For $B > 2$, $\langle\Gamma\rangle \approx 1/N$ indicating at least a weak localization degree induced by long-range correlations.

Fig. 1, our numerical calculation of $\langle\Gamma\rangle$ indicates that $\langle\Gamma\rangle \approx 1/N$ for $B > 2$. In Fig. 3 we complete our analysis by solving numerically the wave equation for an initial pulse $\Psi_0(t) = \sum_{\omega_n<10} \cos(\omega_n t)$ and compute the intensity spectrum $A(\omega)$. As shown in Fig. 3, all the modes with $\omega > \omega_c$ decay, and the medium behaves as a filter to transmit only the modes below frequency $\omega_c \approx 3$. We compute the intensity spectrum $A(\omega)$ by using another kind of incident wave (e.g. $\Psi_0(t) = \exp[-(t - t_0)^2/2\sigma_t^2] \cos(\omega t)$ with $\sigma_t = (1/\sigma_\omega) = 20$ and ω within $[0, 10]$) and no qualitative change in the physical properties was found. Then the numerical evidence reported here, obtained by using TMF and numerical solutions of wave equations, suggests that the low-frequency modes in a 1D chaotic media with long-range correlations could be

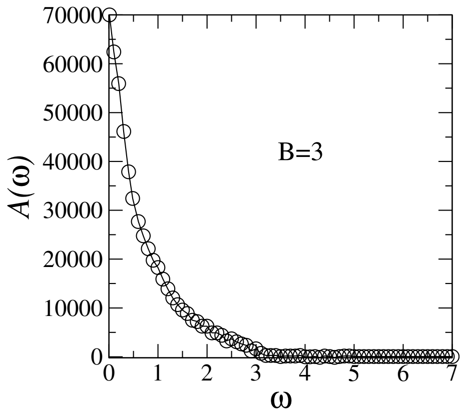

Fig. 3. Intensity spectrum $A(\omega)$ computed by solving numerically the wave equation for an initial pulse $\Psi_0(t) = \sum_{\omega_n<10} \cos(\omega_n t)$. All acoustic modes with $\omega > \omega_c$ decay, and the medium behaves as a filter to transmit only the modes below frequency $\omega_c \approx 3$.

delocalized. However, we need to be careful about the nature extended/localized of these acoustic waves. Both calculations presented until now were done using finite chains. Furthermore, we must note that the zero Lyapunov exponent does not always mean the extended states. The power-law localized states also have the zero Lyapunov exponents. Therefore we need to apply a finite-size scaling procedure to conclude about the nature of low-frequency modes. In Fig. 4 we plot the scaled average Lyapunov exponent $\langle\Gamma\rangle N$ versus N for $B = 2.5$ and 3. Let us stress that for extended states $\langle\Gamma\rangle N$ should be proportional to a constant. From the other side, $\langle\Gamma\rangle N \propto N$ for localized case. $\langle\Gamma\rangle N \propto N^{\nu}$ with $\nu < 1$ signs power-law localized states. Our calculations have clearly shown that the $\langle\Gamma\rangle N$ increase as the system size is

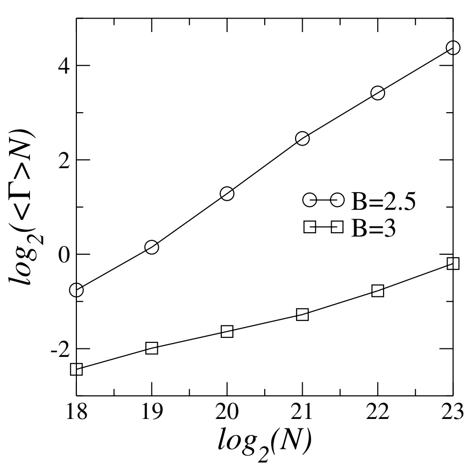

Fig. 4. Scaled average Lyapunov exponent $\langle\Gamma\rangle N$ versus N for $B = 2.5$ and 3. $\langle\Gamma\rangle N$ increases with the system size thus indicating the absence of extended states at the thermodynamic limit.

increased thus indicating the absence of extended states at the thermodynamic limit. Within our numerical precision $\langle\Gamma\rangle N \propto N^{\alpha(B)}$ where $\alpha(B)$ decrease as B is increased. In fact the chaotic distribution of elasticity weakens the degree of localization however does not promote truly extended states.

5. Summary and Conclusion

We studied the propagation of acoustic waves in a 1D chaotic media with a long-range correlated elasticity distribution. The elasticity distribution is generated following the modified Bernoulli map. The map can generate a sequence with a power spectrum $S(f) \sim 1/f^{(2B-3)/(B-1)}$, where B is the single parameter that defines the Bernoulli map. By using a transfer-matrix method we computed the localization length of the allowed acoustic waves. In addition, we have solved directly the scalar wave equation for the propagation of an acoustic wave-packet. Our results have shown that for $B > 2$, the localization length in the low-frequency region ($\omega < \omega_c$) becomes large, however not proportional to the system size. Therefore, the chaotic distribution of elasticity weakens the degree of localization and can promote transport in finite systems. However, this model does not contain truly extended states. We expect that the present work will stimulate further theoretical and experimental investigations along this line.

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