Fractality of largest clusters and the percolation transition in power-law diluted chains

S. S. Albuquerque, F. A. B. F. de Moura, and M. L. Lyra
Departamento de Física, Universidade Federal de Alagoas, Maceió-AL, 57072-970, Brazil

A. J. F. de Souza
Departamento de Física e Matemática, Universidade Federal Rural de Pernambuco, 52171-030 Recife PE, Brazil
(Received 20 December 2004; published 18 July 2005)

Relying on the fractal character of the largest clusters at criticality, we employ a finite-size scaling analysis to obtain an accurate phase-diagram of the percolation transition in chains with bond concentration decaying as a power-law on the form $1/r^{1+\sigma}$. For the particular case of $\sigma=1$, no percolation transition is observed to occur at a finite dilution, in contrast with the finite temperature Kosterlitz-Thouless transition exhibited in Ising and Potts chains with inverse square-law couplings. The fractal dimension of the critical percolation cluster is found to follow distinct dependencies on the decay exponent being numerically fitted by $d_f=0.35+4\sigma/5$ for $0<\sigma<1/2$ and $d_f=(1+\sigma)/2$ for $1/2<\sigma<1$. We also compute average mass ratios of the two largest clusters at criticality.

DOI: 10.1103/PhysRevE.72.016116 PACS number(s): 05.70.Fh, 64.60.Ak, 05.70.Jk, 64.60.Fr

I. INTRODUCTION

The phase transition occurring in systems with power-law decaying interactions is a subject of long-standing interest due to its unique critical behavior. For d-dimensional systems with a n-component order parameter and long-range interactions decaying as $1/r^{d+\sigma}$, renormalization group results as power series expansion in $\epsilon=2\sigma-d$ established that the critical exponents assume d and n-independent values (usually named "classical" values) for $0<\sigma<d/2$ given by $\eta=2-\sigma$, $\nu=1/\sigma$ for the correlation function and correlation length, respectively [1]. For $\sigma>2$ short-range exponents apply. In the intermediate regime of $d/2<\sigma<2$, the exponents depend on the lattice and order parameter dimensionality (nonclassical values) and series expansions up to second order in ϵ have been provided. An interesting feature is that the correlation function exponent η seems to stick on its classical value. This has been conjectured to be valid to all orders in ϵ, although demonstrated only up to second order. In addition to the theoretical interest in the rich critical behavior depicted by systems with power-law decaying interactions, long-range forces and couplings have been used to describe several interesting physical phenomena such as spin-glass transitions [2], criticality in highly ionic systems [3], the Kondo problem [4], Casimir-like effects in strongly correlated systems [5,6], neural systems modeling [7], negative specific heat in a Lennard-Jones-like gas [8], and complex networks [9].

In the particular case of one-dimensional (1D) models with long-range interactions, there is no transition if $\sigma>1$. Therefore, the regime of nonclassical critical behavior is restricted to the range $1/2<\sigma<1$. At $\sigma=1/2$ the critical behavior exhibits logarithmic corrections to the classical one. The opposite case of $\sigma=1$ has been the subject of intense investigations [10–13]. In models where the local variables have a continuum of states, the critical temperature vanishes continuously as $\sigma\to 1$ from below [14]. On the other hand, 1D models with inverse square-law interactions and local variables having a finite number of states display an order-disorder transition at a finite temperature. This transition has been shown to belong to the class of topological phase transitions similar to the Kosterlitz-Thouless (KT) one exhibited by the XY two-dimensional model [10]. The correlation length has an exponential divergence as approaching the critical temperature and the order parameter jumps discontinuously to zero. The q-state Potts model is a very instructive example where a KT transition is expected to occur for any $q\geqslant 2$. Although a preliminary real space renormalization group result indicated that the transition temperature T_c should be the same for any value of q [15], more recent numerical simulations have shown that T_c is a decreasing function of the number of states q [13]. For slower decaying interactions ($\sigma<1$), the transition can become a first order one [16], with the threshold value $\sigma_c(q)<1$ for all q [17].

The corresponding counterpart model to study the geometric percolation transition in randomly diluted systems with power-law decaying bond concentration $p(r)\propto 1/r^{1+\sigma}$ was introduced more than two decades ago [18]. It was initially conjectured that an infinite cluster would be present for any finite bond concentration in the case of $\sigma<1$ [18]. This conjecture was readily shown to fail and that no percolation cluster could exist for $p<1/2\zeta(\sigma)$, where $\zeta(\sigma)$ is the Reimann zeta function [19]. The existence of a transition between a percolating and a nonpercolating regime was demonstrated for $\sigma<1$ [20]. In the limiting case of $\sigma=1$, it was argued the occurrence of an intermediate phase with slowly decaying correlations and a discontinuity of the percolation density [12,21]. More recently, the scaling behavior of the shortest-path distance [22,23] and of the random walk problem [24] in long-range percolation have being analyzed. However, except by the naive numerical estimate by Rego *et al.* [25], no accurate phase diagram for the percolation transition in this model has been obtained so far.

In this work, we will exploit the fractal scaling behavior of the largest clusters at the percolation threshold to obtain precise estimates of the critical point of the long-range 1D percolation problem. The scaling analysis will be based on

the existence of effectively many infinite clusters at the transition, all of them sharing the same fractal dimension [26,27]. This feature allows for the use of average mass ratios as zero-exponent scaling variables to locate the critical point from simulations on relatively small lattices [28]. This strategy was successfully applied to the percolation problem in d-dimensional hypercubic lattices providing accurate estimates of the percolation threshold up to $d=7$ as well as several relevant mass distribution functions associated with the two largest clusters [29]. Here, besides providing the critical line $p_c(\sigma)$, we will also report on the critical average mass ratios of the second largest and largest clusters $\langle M_2/M_1\rangle, \langle M_2\rangle/\langle M_1\rangle$, as well as on the fractal dimension of the largest cluster $d_f(\sigma)$. At extremal values of the power-law decay exponent σ, an alternative numerical renormalization group will be used to refine the critical parameters estimates.

II. LONG-RANGE PERCOLATION MODEL AND FRACTAL SCALING

We will consider the 1D long-range percolation model as a closed chain with L sites. For each pair of sites (i,j), the probability of having a bond connecting them is given by $p(i,j)=p/r^{1+\sigma}$, where r is the smallest distance between this pair of sites and p is the first-neighbors bond concentration. When only short-range connections are allowed, an infinite cluster containing a finite fraction of the chain does not exist for any $p<1$. This picture remains valid whenever the power-law exponent $\sigma>1$. For $\sigma<1$ an infinite cluster emerges above a critical concentration $p_c(\sigma)$ due to the high connectivity induced by the long-range couplings. For $\sigma<0$ there will be a spanning cluster with a finite density for any nonvanishing value of p. Therefore, a continuous percolation transition can occur only in the regime of $0<\sigma<1$.

An extended version of the above percolation problem was studied in Refs. [12,19–21] on which $p(i,j)=\beta/r^{1+\sigma}$ for $r>1$ while p refers only to the first neighbors bond concentration. In the region of $0<\sigma<1$ the extended model also presents a continuous percolation transition. For the limiting case of $\sigma=1$, it has been rigorously proved that there is a discontinuity in the infinite-cluster density occurring at $\beta_c(p)>1$, with β_c approaching to unity as $p\to 1$. The presently studied percolation problem corresponds to the particular case $\beta=p\leqslant 1$ of the above extended model and, as such, only can fully probe the continuous transition that takes place in the regime of $0<\sigma<1$. However, we will also investigate the $\sigma=1$ case aiming to characterize the scaling behavior in the nonpercolating regime at the vicinity of the asymptotic discontinuous percolation transition which sets up as $p\to 1$. In what follows we will continue referring to p as the first neighbors bond concentration, but it shall be kept in mind that it affects the strength of all bonds.

The distribution of cluster masses at the percolation threshold has usually been explored in order to obtain new properties at the critical point [30–33]. An interesting feature is observed when the clusters formed at criticality are ranked in decreasing order of size. In the regime of large system sizes, the average mass of the kth largest cluster scale as $\langle M_k\rangle \propto L^{d_f}$, with the same fractal dimension for any rank [26,27,30,34]. Specifically, the second largest cluster is proportional in mass to the largest cluster at the percolation threshold. The average ratio between the masses of the largest and second largest clusters has been shown to be scale invariant at criticality, with its value depending on the lattice geometry and boundary conditions [28–30,35].

According to the finite size scaling hypothesis, the average size of the largest cluster near the percolation threshold scales as $\langle M_1\rangle \sim L^{d-\beta/\nu}$. The same scaling behavior stands for $\langle M_2\rangle$ [26,30]. As a consequence, the average mass ratio $\langle M_2/M_1\rangle$ behaves as a zero exponent critical quantity scaling as

$$\langle M_2/M_1\rangle = f[(p-p_c)L^{1/\nu}] \tag{1}$$

with the scaling function $f(-\infty)=1$ and $f(+\infty)=0$. A similar scaling relation also holds for $\langle M_2\rangle/\langle M_1\rangle$. According to the above scaling behavior these ratios are size independent at the critical percolation threshold. Below the percolation threshold, the two largest clusters are similar in size and the above mass ratios approach to unity as the system size increases. On the other hand, the largest cluster predominates above the critical concentration and the mass ratio vanishes in the infinite size limit. Therefore, curves obtained from distinct system sizes of either one of the above cluster mass ratios will intercept at a common point when plotted as a function of the bond concentration [28,29].

As the coupling power-law exponent $\sigma\to 0$, the number of connecting bonds is reduced (although with a longer-range character). In this regime, the incipient percolation clusters have smaller sizes. Therefore, finite-size scaling analysis based on the joint properties of the largest and second largest clusters shall become less confident than one based solely on the scaling of the largest cluster. This difficulty reflects the larger corrections to scaling presented by higher ranked clusters. To refine the estimate of critical parameters in this regime, a numerical renormalization group can be used based on the behavior of the following set of auxiliary functions:

$$g(L,L',p) = \ln[M_1(L,p)/M_1(L',p)]/\ln(L/L'). \tag{2}$$

These are derived solely from the size of the largest cluster and, according to the finite-size scaling hypothesis, shall intercept at a common point $g(L,L',p_c)=d_f$ for any pair of sizes (L,L'), with a possible small spread due to corrections to scaling. This numerical renormalization group scheme is also expected to be more accurate at very low dilutions due to the reduced size of the second largest cluster in this regime. The accuracy of such phenomenological renormalization based on Monte Carlo data was firstly demonstrated by investigating site percolation in a simple cubic lattice [36].

III. FINITE-SIZE SCALING: PHASE DIAGRAM AND FRACTAL DIMENSION

In our simulations, we considered chain sizes ranging from $L=200$ up to $L=3200$. For a given value of the decay exponent σ, we measured the average value (over 40 000 random bond distributions) of the size of the two largest clusters and the average ratio between their sizes as a function of the first neighbors concentration p.

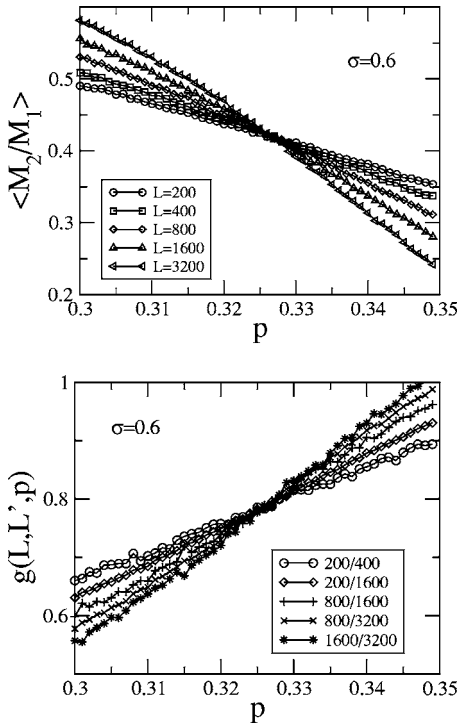

FIG. 1. (a) $\langle M_2/M_1\rangle$ and (b) $g(L,L',p)$ versus p for $\sigma=0.6$. Data were obtained after a configurational average over 40 000 samples. Error bars are much smaller than symbols size. The common point of distinct curves in either (a) or (b) determines p_c. The scale invariant average mass ratio at percolation can also be extracted from (a) whereas the fractal dimension of the critical percolation cluster is given by $g(L,L',p_c)$ from (b).

The critical first neighbors concentration $p_c(\sigma)$ was obtained from the interception point of the average mass ratio $\langle M_2/M_1\rangle$ measured for distinct chain sizes, as well as from the interception point of the set of auxiliary functions $g(L,L',p)$ obtained from the numerical renormalization analysis of the largest cluster size. A representative plot is shown in Fig. 1 for the particular case of $\sigma=0.6$. Both methods provide roughly the same estimate for the critical concentration $p_c(\sigma=0.6)=0.327(1)$. In either case, the interception point exhibits a very small spread which indicates that corrections to scaling are negligible for the system sizes simulated at this specific value of the power-law decay exponent σ.

The above analysis was employed to obtain the critical line $p_c(\sigma)$ for the full range $0<\sigma<1$. Our results are depicted in Fig. 2. As expected, the estimates are slightly distinct for extremal values of σ. The one obtained solely from the scaling behavior of the largest cluster mass is to be considered the more reliable in these regimes due to larger corrections to scaling presented by the size of the second largest cluster at weak and strong dilutions. We obtained that the critical concentration continuously grows towards $p_c(1)=1$. Therefore, the largest cluster remains finite for any first-neighbor dilution in the case of square-law decaying bond concentrations. This fact is consistent with previous analytical arguments demonstrating that a percolating phase can only be observed in the extended long-range percolation

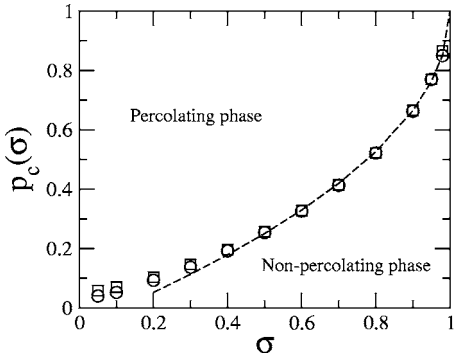

FIG. 2. The estimated phase diagram for the long-range percolation model with power-law decaying bond concentrations. Critical points were estimated from the scaling behavior of the average cluster mass ratio (squares) as well as from the numerical renormalization of the largest cluster mass (circles). The dashed line represents the curve $1-p_c(\sigma)\propto\sqrt{1-\sigma}$ which fits the data well especially for σ close to unity.

model for $\beta\geqslant 1$ [21]. It is worth mentioning that the estimated data for the critical concentration follows closely a simple power law $1-p_c(\sigma)\propto\sqrt{1-\sigma}$ (dashed line in Fig. 2), particularly for σ close to unity.

In order to characterize the scaling behavior of the bordering case $\sigma=1$, we analyzed the finite-size dependence of the normalized average size of the largest cluster at the vicinity of $p=1$. We found that, for a weak dilution of first-neighbors couplings, $\langle M_1\rangle/L$ decreases very slowly as the chain size increases, indicating a slow logarithmic convergence to the thermodynamic limit. Indeed, our data from distinct chain sizes were collapsed into a universal scaling function in the form

$$\langle M_1\rangle/L = f[(1-p)(\ln L)^{1/2}] \tag{3}$$

as shown in Fig. 3. This scaling law supports the absence of an infinite cluster at finite dilutions in the limit $L\to\infty$, i.e., $\langle M_1(p\neq 1, L\to\infty)\rangle/L\to 0$. On the other hand, $\langle M_1(p=1, L\to\infty)\rangle/L=1$. A discontinuity on the percolation density at p

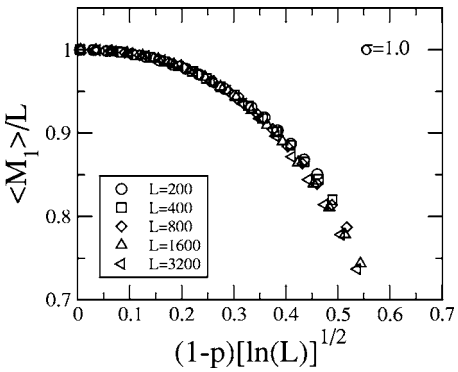

FIG. 3. Collapse of data from distinct chain sizes for the bordering case $\sigma=1$. The normalized largest cluster size follows a universal function of $(1-p)(\ln L)^{1/2}$, thus indicating the absence of an infinite cluster at finite dilutions and the exponential growth of the average largest cluster size as $p\to 1$.

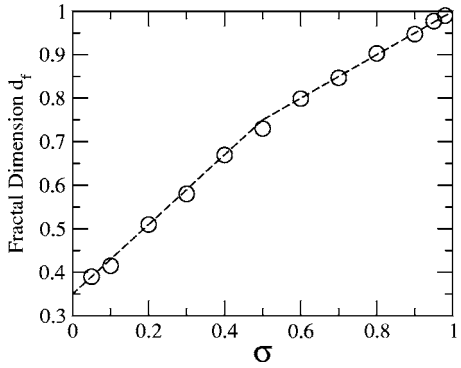

FIG. 4. The estimated fractal dimension of the critical percolation cluster as a function of the decay exponent σ. For $\sigma > 1/2$ the data are well fitted by $d_f(\sigma) = (1+\sigma)/2$, which is similar to the classical behavior predicted by the renormalization group for systems with power-law decaying interactions. For slower decaying bond concentrations, there are deviations from this relation. The numerical trend points towards $d_f(\sigma < 1/2) = 0.35 + 4\sigma/5$. The dashed broken line represent the above linear relations.

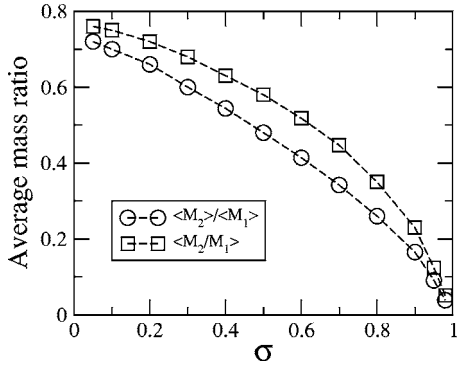

FIG. 5. The critical average mass ratios $\langle M_2/M_1 \rangle$ and $\langle M_2 \rangle / \langle M_1 \rangle$ and a function of the decay exponent σ. They approach to very small values at the border of the long-range regime as the critical percolation clusters become dense at the onset of the short-range regime.

$=1$ is natural to occur even in the nearest-neighbor percolation model, although in this case the scaling form is trivial [37]. The above finite-size scaling indicates a nontrivial Kosterlitz-Thouless-like transition. Therefore, the reported discontinuity of the percolation density at $p=1$ is reminiscent of the discontinuous percolation transition taking place in the extended long-range percolation model at $\beta = p = 1$ [21]. The logarithmic scaling variable reveals that the average largest cluster size grows exponentially as $p \to 1$ once the characteristic cluster size scale as $e^{(1-p)^2}$. This last feature is consistent with a diverging correlation length exponent for this asymptotic percolation transition.

The fractal dimension d_f of the critical percolating cluster was measured from the numerical renormalization scheme. The σ dependence of d_f is reported in Fig. 4. It is particularly interesting to notice that, for $\sigma > 1/2$, it is well fitted by $d_f(\sigma) = (1+\sigma)/2$ within our numerical accuracy. Using the scaling relations $d_f = d - \beta/\nu$ and $2 - \eta = d - 2\beta/\nu$, the above σ dependence of d_f implies $\eta = 2 - \sigma$. This is the classical relation predicted by renormalization group calculations for systems with power-law decaying interactions [1,38]. However, the pure geometric percolation transition here studied deviates from this classical relation for slowly decaying bond concentrations. For $\sigma < 1/2$, the fractal dimension appears to follow a distinct linear dependence $d_f = 0.35 + 4/5\sigma$. The case $\sigma = 1/2$ shows a small deviation from either of the above trends, which may be due to logarithmic corrections to scaling not considered in the present study.

We have also measured the critical average mass ratios $\langle M_2/M_1 \rangle$ and $\langle M_2 \rangle / \langle M_1 \rangle$ (see Fig. 5). For slowly decaying bond concentrations at which the critical state is strongly diluted, the largest and second largest clusters at criticality are very similar in size. On the other hand, close to the border case $\sigma = 1$, the critical state is only weakly diluted and the largest cluster is much larger than the second largest one, although they share the same fractal dimension. The fact that $\langle M_2/M_1 \rangle > \langle M_2 \rangle / \langle M_1 \rangle$ reflects the fluctuations of the cluster mass ratio over distinct bond distributions.

IV. CONCLUSIONS

In summary, we have reported a detailed finite-size scaling analysis of the percolation transition on linear chains with power-law decaying bond concentrations $p(r) = p/r^{1+\sigma}$. Exploring the fractal behavior of the largest and second largest clusters at the percolation threshold, we obtained accurate estimates for the critical percolation threshold and cluster mass fractal dimension for the range $0 < \sigma < 1$. We found that $p_c(\sigma = 1) = 1$ and, therefore, no infinite cluster exists in the thermodynamic limit for square-law decaying bond concentrations for any finite first-neighbor dilution. Our finite size scaling analysis at the vicinity of $p_c(\sigma = 1) = 1$ indicates that, while the nature of the transition in the present percolation model is still KT-like, there is also some difference between our geometrical model and the finite-temperature KT transitions exhibited by 1D thermal systems with $1/r^2$ interactions. However, it is consistent with the predicted discontinuity of the percolation density [21]. For the regime $0 < \sigma < 1$, we determined the critical average ratio between the masses of the two largest clusters. Close to the border case of $\sigma = 1$ the largest cluster becomes much larger than the second largest, resulting in a slow convergence to the thermodynamic behavior as the onset of the short-range behavior approaches. The fractal dimension of the critical percolation cluster was found to exhibit two distinct regimes for σ above and below 1/2. For $\sigma > 1/2$ it was found to coincide with the *classical* prediction for the critical exponents of systems with power-law decaying interactions, although these are not, a priori, expected to apply for the pure geometric transition. It would be valuable to have these two regimes reproduced by renormalization group arguments tailored for this long-range percolation problem. The present finite-size scaling analysis can also be employed to characterize the rich phase diagram depicted by the extended version of the long-range 1D percolation problem with distinct parameters for the short- and long-range connections. This study can bring important new information about the critical behavior, especially for the border case of square-law decaying bond concentrations. We hope to report on these questions in a future communication.

ACKNOWLEDGMENTS

We would like to acknowledge the partial financial support provided by the Brazilian research agencies CNPq and CAPES and by the Alagoas state research agency FAPEAL. A.J.F.S. also acknowledges FACEPE (Pernambuco State research agency) for partial funding.

[1] M. E. Fisher, S.-K. Ma, and B. G. Nickel, Phys. Rev. Lett. **29**, 917 (1972).
[2] P. J. Ford, Contemp. Phys. **23**, 141 (1982).
[3] K. S. Pitzer, M. C. P. de Lima, and D. R. Schreiber, J. Phys. Chem. **89** 1854 (1985).
[4] P. W. Anderson and G. Yuval, Phys. Rev. B **1**, 1522 (1970).
[5] M. L. Lyra, M. Kardar, and N. F. Svaiter, Phys. Rev. E **47**, 3456 (1993).
[6] T. W. Burkhardt and E. Eisenriegler, Phys. Rev. Lett. **74**, 3189 (1995).
[7] D. J. Amit, *Modeling Brain Functions* (Cambridge University Press, Cambridge, 1989).
[8] E. P. Borges and C. Tsallis, Physica A **305**, 148 (2002).
[9] D. J. Watts and S. H. Strogatz, Nature (London) **393**, 440 (1998).
[10] J. L. Cardy, J. Phys. A **14**, 1407 (1981).
[11] M. Aizenman, J. T. Chayes, L. Chayes, and C. M. Newman, J. Stat. Phys. **50**, 1 (1988).
[12] J. Z. Imbrie and C. M. Newman, Commun. Math. Phys. **118**, 303 (1988).
[13] E. Luijten and H. Meßingfeld, Phys. Rev. Lett. **86**, 5305 (2001).
[14] J. M. Kosterlitz, Phys. Rev. Lett. **37**, 1577 (1976).
[15] S. A. Cannas and A. C. N. de Magalhes, J. Phys. A **30**, 3345 (1997).
[16] Z. Glumac and K. Uzelac, Phys. Rev. E **58**, 4372 (1998).
[17] S. Reynal and H. T. Diep, Phys. Rev. E **69**, 026109 (2004).
[18] Z. Q. Zhang, F. C. Pu, and B. Z. Li, J. Phys. A **16**, L85 (1983).
[19] L. S. Schulman, J. Phys. A **16**, L639 (1983).
[20] C. M. Newman and L. S. Schulman, Commun. Math. Phys. **104**, 547 (1986).
[21] M. Aizenman and C. M. Newman, Commun. Math. Phys. **107**, 611 (1986).
[22] C. F. Moukarzel and M. A. de Menezes, Phys. Rev. E **65**, 056709 (2002).
[23] M. Biskup, Ann. Prob. **32**, 2938 (2004).
[24] N. Berger, Commun. Math. Phys. **226**, 531 (2002).
[25] H. H. A. Rego, L. S. Lucena, L. R. da Silva, and C. Tsallis, Physica A **266**, 42 (1999).
[26] N. Jan, Physica A **266**, 72 (1999).
[27] D. Stauffer, Int. J. Mod. Phys. C **11**, 519 (2000).
[28] C. R. da Silva, M. L. Lyra, and G. M. Viswanathan, Int. J. Mod. Phys. C **11** 1411 (2000)
[29] C. R. da Silva, M. L. Lyra, and G. M. Viswanathan, Phys. Rev. E **66**, 056107 (2002).
[30] N. Jan, D. Stauffer and A. Aharony, J. Stat. Phys. **92**, 325 (1998).
[31] M. Aizenman, Nucl. Phys. B **485**, 551 (1997).
[32] M. S. Watanabe, Phys. Rev. E **53**, 4187 (1996).
[33] P. Sen, Int. J. Mod. Phys. C **8**, 229 (1997); **10**, 747 (1999).
[34] S. Macleod and N. Jan, Int. J. Mod. Phys. C **9**, 289 (1998).
[35] A. Margolina and H. J. Herrmann, Phys. Lett. **104A**, 295 (1984).
[36] M. Sahimi, J. Phys. A **18**, 3597 (1985).
[37] D. Stauffer and C. Jayaprakash, Phys. Lett. **64A**, 433 (1978).
[38] W. K. Theumann and M. A. Gusmão, Phys. Rev. B **31**, 379 (1985).